Synchronization and Consensus in Complex Networks Involving Discontinuity, Switching and Arbitrary Weighting

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Consider the following coupled networks:

$$\frac{d x^i}{dt} = f(x^i, t) + \sum_{j=1}^{N} a_{ij} \Gamma x^j, \ i = 1, \cdots, N, \tag{1}$$

where $f: \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}^n$, $A = [a_{ij}]_{i,j=1}^{N}$ is the adjacent matrix, $\Gamma = \text{diag}[^{\gamma_1, \cdots, \gamma_n}]$ is the inner coupling matrix.
We assume that $f$ is a measurable map, which may be non-Lipschitz or even discontinuous. The discontinuous set of $f$ is a set of countable hypersurfaces with lower dimension. When $f$ is discontinuous, the solution is understood in the Filippov sense[1]. That is, define

$$
\mathcal{K}[f](x, t) = \bigcap_{\delta > 0} \bigcap_{\mu(S) = 0} \text{co} f(B(x, \delta) \setminus S, t),
$$

then a Filippov solution of (1) is defined as a solution of the differential inclusion:

$$
\dot{x} \in \mathcal{K}[f](x, t), \quad a.e. \ t \in [t_0, t_1].
$$

Denote $F(x, t) = \mathcal{K}[f](x, t)$, we assume that the map $f$ satisfies the following condition $\lambda$:

- $F(x, t)$ satisfies the basic conditions;
- For any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, $\|F(x, t)\| \leq a(t)\|x\| + b(t)$, where $a(t), b(t)$ are functions defined on $\mathbb{R}^+$ which are integrable on any finite interval of $t$;
- For any $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, $\|F(x, t) - F(y, t)\| \leq h(\|x - y\|)$, where $h: \mathbb{R}^+ \mapsto \mathbb{R}^+$ is continuous and nondecreasing;
\( B(S_i, \delta) \) and \( B(S_j, \delta) \) have no intersection for all \( i \neq j \) and some \( \delta > 0 \).

On the closure of each connected component of continuous region, there exists a continuous extension of \( f \);

On each discontinuous hypersurface \( S_i = \{ (y, t) | \varphi^i(y, t) = 0 \} \), \( \nabla \varphi^i(y, t) \neq 0 \) for almost all \( t \), where
\[
\nabla \varphi^i(y, t) = \left[ \frac{\partial \varphi^i}{\partial y_1}, \cdots, \frac{\partial \varphi^i}{\partial y_n} \right].
\]

From the Filippov theory, if \( f \) satisfies condition \( \lambda \), the Filippov solution exists and can be extended to \( \mathbb{R}^+ \).
**Definition ([2])**

Let $M_{m \times n}(\mathbb{R})$ be the class of $m \times n$ real matrices, and $T_1(k, K)$ be the set of matrices with entries in $M_{k \times k}(\mathbb{R})$ such that the sum of the entries in each row is equal to $K$ for some $K \in M_{k \times k}(\mathbb{R})$. Let $M_2(k)$ be the set of matrices $M$ with entries in $\{\alpha I_k : \alpha \in \mathbb{R}\}$ such that each row of $M$ contains zeros and exactly one $\alpha I_k$ and one $-\alpha I_k$ for some nonzero $\alpha$, and for any pair of indices $i$ and $j$ there exist indices $i_1, i_2, \ldots, i_l$ with $i_1 = i$ and $i_l = j$ such that for all $1 \leq q < l, M_{p,i_q} \neq 0$ and $M_{p,i_q+1} \neq 0$ for some integer $p$.

Let $C \in M_2(k)$ be an $m \times n$ matrix, and $A \in T_1(k, K)$ be an $n \times n$ matrix. Define $B = S(C, A) = CAGC$ such that $CA = BC$.

Theorem 1

Suppose that \( f(x, t) \) satisfies condition \( \lambda \). If for any given initial value \( \tilde{x}_0 \in \mathbb{R}^{nN} \), letting \( \tilde{x}(t) = [(x^1(t))^\top, (x^2(t))^\top, \cdots, (x^N(t))^\top]^\top \) be the solution from \( \tilde{x}_0 \), there exist a \((N - 1) \times N\) matrix \( M \in M_2(n) \), a \( N \otimes N\) matrix \( T \in T_1(n, K) \), and a positive definite matrix \( P \in M_{nN \times nN}(\mathbb{R}) \) such that the symmetric part of \( PS(M, T + A \otimes \Gamma_n) \) is negative definite and

\[
\sup_{t \geq 0} \left\{ \int_0^t \sup \left\{ \tilde{x}(s)^\top M^\top P M [\tilde{F}(\tilde{x}(s), s) - T\tilde{x}(s)] \right\} ds \right\} < +\infty,
\]

where \( \tilde{F}(\tilde{x}, t) = [F(x^1, t)^\top, \cdots, F(x^N, t)^\top]^\top \), then we have

\[
\lim_{t \to +\infty} \|x^i(t) - x^j(t)\| = 0, \quad i, j = 1, 2, \cdots, N.
\]
In particular, let \( M = \begin{bmatrix} -I_n & I_n & 0 \\ \vdots & \ddots & \ddots \\ -I_n & 0 & I_n \end{bmatrix}, \Gamma = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \)

with \( n_1 \leq n \), \( P = I_N \otimes I_n \) and \( T = I_N \otimes \Delta \) with

\[
\Delta = \begin{bmatrix} LI_{n_1} & 0 \\ 0 & -\alpha I_{n-n_1} \end{bmatrix}
\]

for some constants \( L \in \mathbb{R} \) and \( \alpha > 0 \).

Then, we have

\[
S(M, T + A \otimes \Gamma) = (LI_{N-1} + B) \otimes \Gamma - \alpha I_{N-1} \otimes \begin{bmatrix} 0 & 0 \\ 0 & I_{n-n_1} \end{bmatrix},
\]

where \( B = [b_{ij}]_{N-1 \times N-1} \) with \( b_{ij} = a_{i+1,j+1} - a_{1,j+1} \).
Corollary 2

Suppose that \( f(x, t) \) satisfies condition \( \lambda \), \( \Delta \) is defined as above. If there exist \( C \in \mathbb{R}, L < -\lambda_{\text{max}}(B^S) \) and \( \alpha > 0 \) such that

\[
\sup_{t \geq 0} \left\{ \int_0^t \sup \left\{ [F(x^i(s), s) - F(x^1(s), s) - \Delta(x^i(s) - x^1(s))]^T [x^i(s) - x^1(s)] \right\} ds \right\} \leq C
\]

holds for all \( i = 2, 3, \cdots, N \), then the coupled system (1) synchronizes. i.e., \( \lim_{t \to +\infty} \|x^i(t) - x^j(t)\| = 0, \forall i, j = 1, 2, \cdots, N \).
Let $\Delta$ be defined as in Corollary 2 and denote

$$\tilde{T}_i = \left\{ t \in \mathbb{R}^+ \mid \sup \left\{ \left[ F(x^i(t), t) - F(x^1(t), t) - \Delta(x^i(t) - x^1(t)) \right]^T [x^i(t) - x^1(t)] \right\} > 0 \right\}$$ (5)

and $T_i = \mathbb{R}^+ \setminus \tilde{T}_i$.

**Corollary 3**

Suppose that $f(x, t)$ satisfies condition $\lambda$. Given an initial value $\tilde{x}_0 \in \mathbb{R}^{nN}$, if the trajectory $[x^1T(t), \cdots, x^{NT}]^T$ satisfies that $\|x^i(t) - x^1(t)\|$ is uniformly bounded (the bound depends on the initial value but is uniform for all $t \in \mathbb{R}^+$) and $\mu(\tilde{T}_i) < +\infty$ for $i = 2, 3, \cdots, N$, then system (1) synchronizes.
Consider $N$ switching systems with $K$ switching dynamics:

$$\dot{y}^i = f_k(y^i, t), \text{ if } \xi^i \in \Omega_k, \ k = 1, \cdots, K.$$  \hfill (6)

Here, $\xi^i$ denotes the switching signal of the $i$-th system, which can be described by a differential equation:

$$\dot{\xi}^i = g(\xi^i, t), \ i = 1, \cdots, N,$$

where $\Omega_k, \ k = 1, \cdots, K$, denote domains on the space of the signals.
Under the assumption that on each $\Omega_k$, the map $f_k$ is contracting, we present a coupling scheme over a graph $G$ with the Laplacian $A = [a_{ij}]_{i,j=1}^{N}$ via only coupling their switching-driving signals as follows:

$$\begin{cases} 
\dot{\xi}^i = g(\xi^i, t) + \sum_{j=1}^{N} a_{ij} \xi^j \\
\dot{y}^i = f_k(y^i, t), \ i \text{ if } \xi_k \in \Omega_k, \ k = 1, \cdots, K,
\end{cases} \quad (7)$$

where $i = 1, \cdots, N$. 
Consider a network of $N$ all-to-all coupled 3-dimensional individual dynamical systems that has the form (6) where

$$f_1(x, t) = \cos(t)$$

and

$$f_k(x, t) =
\begin{cases}
-\frac{3}{2}x_k + |\sin(t)|x_{k+1} + |\cos(t)|x_{k+2} - 1 & x_1 < 0 \\
-\frac{3}{2}x_k + |\sin(t)|x_{k+1} + |\cos(t)|x_{k+2} + 1 & x_1 > 0,
\end{cases}$$

for $k = 2, 3$, and $x_4 = x_2, x_5 = x_3$. 

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Figure: Trajectory of an isolated node

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Figure: The trajectory of $x_1^i$
Figure: The trajectory of $x^i_2$
Figure: The trajectory of $x_3^i$
Figure: The synchronization error
Let $\bar{\Omega} = \{ A = [a_{ij}]_{i,j=1}^n | \sum_{j=1}^n a_{ij} = c_A, i = 1, \ldots, n \}$, consider the following random discrete-time dynamical linear system:

$$x(k + 1) = A_k(\omega)x(k)$$

(8)

where $k = 1, 2, \cdots$ is the discrete time index, $x(k) \in \mathbb{R}^n$ is the state vector at time $k$, $\omega = (\omega_1, \omega_2, \cdots) \in \Omega = \Omega_1 \times \Omega_2 \times \cdots$ and $A_k(\omega) = \omega_k \in \Omega_k \subset \bar{\Omega}$. 
Define a function $\eta(A)$ as

$$
\eta(A) = \frac{1}{2} \max_{i,j,l,m} \left\{ a_{li} - a_{mi} - a_{lj} + a_{mj} + \sum_{k \neq i,j} |a_{lk} - a_{mk}| \right\}
$$

(9)

**Theorem 4**

If $\mathbb{P}(\{\omega \in \Omega : \prod_{k=1}^{+\infty} \eta(\omega_k) = 0\}) = 1$, the system (8) will achieve consensus almost surely, i.e., for any initial value $x(1) \in \mathbb{R}^n$, the solution of (8) $x(t,\omega)$ satisfies

$$
\mathbb{P}(\{\omega : \lim_{k \to +\infty} \max_{i,j} |x_i(k,\omega) - x_j(k,\omega)| = 0\}) = 1
$$

(10)
Assumption 1 (i.i.d.)

- $\Omega_1 = \Omega_2 = \cdots$ are bounded,
- $\mathbb{P} = \mu \times \mu \times \cdots$ for some probability measure defined on $\Omega_1$.

Corollary 5

Under Assumption 1, if the expectation $E \ln \eta(A_1(\omega)) < 0$ (it is possible that $E \ln \eta(A_1(\omega) = -\infty$), then the system (8) will achieve consensus almost surely.
Assumption 2 (Markov)

- $\Omega_1 = \Omega_2 = \cdots$ are compact subsets of $\bar{\Omega}$,
- $\mathcal{P}(A|x)$ are Markov transition probabilities such that there’s a unique probability $\pi$ on $\mathcal{B}_1$ satisfying

$$
\pi(A) = \int \mathcal{P}(A|x)\pi(dx)
$$

for all $A \in \mathcal{B}_1$.

Corollary 6

Under Assumption 2, if $E_{\pi} \ln \eta(A_1(\omega)) < 0$, then system (8) will achieve consensus almost surely, where $E_{\pi}(\cdot)$ is the expectation induced by $\pi$. 

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Figure: Almost sure consensus of 5 agents (discrete-time, i.i.d case)
Let $\Omega^c_0 = \{ \text{set of matrices of order } n \text{ with zero row sum} \}$, consider the following switched linear dynamical system

$$\dot{x}(t) = -L_k(\omega)x(t), \quad t \in [t_{k-1}, t_k),$$

(11)

where $\omega = (\omega_1, \omega_2, \cdots)$ with $\omega_k \in \Omega_k \subset \Omega^c_0$, and $L_k(\omega) = \omega_k$. The switching time sequence $\{t_k\}_{k=0}^{+\infty}$ forms a partition of $[0, +\infty)$ with $0 = t_0 < t_1 < t_2 < \cdots$. 

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For $A = [a_{ij}]_{n \times n}$, define a function $\xi: \Omega_c^0 \rightarrow \mathbb{R}$ as:

$$\xi(A) = \max_{i,j} \{- (a_{ij} + a_{ji}) - \sum_{k \neq i,j} \min(a_{ik}, a_{jk})\}.$$

**Theorem 7**

The switched system (11) will achieve consensus almost surely, if

$$\mathbb{P}(\{\limsup_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \xi(L_k(\omega)) \Delta t_k < 0\}) = 1,$$  \hspace{1cm} (12)$$

$$\mathbb{P}(\{\Delta t_n = +\infty\}) = 0,$$  \hspace{1cm} (13)$$

$$\lim_{n \to +\infty} \frac{\Delta t_n}{n} = 0, \quad a.s. \mathbb{P}.$$  \hspace{1cm} (14)
Assumption 3 (i.i.d.)

- $\omega_1, \omega_2, \cdots$ are i.i.d,
- $\Delta t_1, \Delta t_2, \cdots$ are i.i.d with $E\Delta t_1 < +\infty$ and $E(\Delta t_1 - E\Delta t_1)^2 < +\infty$,
- Each $\omega_k$ and $\Delta t_k$ are independent.

Corollary 8

Under Assumption 3, the switched system (11) will achieve consensus almost surely if

$$E\xi(\omega_1) < 0$$

(15)
Assumption 4 (Markov)

- \( \{\omega_k\}_{k=1}^{+\infty} \) forms a Markov chain,
- \( \mathcal{P}(A|x) \) are Markov transition probabilities such that there’s a unique probability \( \pi \) on \( \mathcal{B}_1 \) satisfying \( \pi(A) = \int \mathcal{P}(A|x)\pi(dx) \) for all \( A \in \mathcal{B}_1 \),
- \( \Delta t_1, \Delta t_2, \cdots \) are i.i.d with \( E\Delta t_1 < +\infty \) and \( E(\Delta t_1 - E\Delta t_1)^2 < +\infty \),
- Each \( \omega_k \) and \( \Delta t_k \) are independent.

Corollary 9

Under Assumption 4, the switched system (11) will achieve consensus almost surely, if \( E_{\pi}\xi < 0 \).
**Figure:** Almost sure consensus in 5 agents (continuous-time, Markov Chains)
Thank You!